Instanton solutions in the problem of wrinkled flame-front dynamics

D. Volchenkov[1,2,*] and R. Lima[2]
[1]Solid State Electronics Department, Research Institute of Physics, St.-Petersburg State University, St. Petersburg, Russia
[2]Centre de Physique Theorique, CNRS, Luminy Case-907, 13288 Marseille, France
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The statistics of the slopes of wrinkling flames propagating through an infinitely wide channel is investigated by the quantum-field-theory methods. We dwell on the WKB approximation in the functional integral, which is analogous to the Wyld functional integral in turbulence. The main contribution to statistics is due to a coupled field-force configuration. This configuration is related to a kink between metastable exact pole solutions of the Sivashinsky equation. These kinks are responsible for both the formation of new cusps and the rapid power-law acceleration of the mean flame front. The problem of asymptotic stability of the solutions is discussed.

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I. INTRODUCTION

It had been shown in Ref. [1] that under a weakly nonlinear approximation, the dynamics of a wrinkled flames propagating through an infinitely wide channel is governed by a nonlinear partial differential equation (PDE)

$$\frac{\partial\Phi}{\partial t}=\frac{1}{2}U_b\left(\frac{\partial\Phi}{\partial x}\right)^2+D_M\frac{\partial^2\Phi}{\partial x^2}+\frac{1}{2}\gamma U_b\Lambda\{\Phi\}. \qquad (1)$$

Here, Φ is the interface of a distorted planar flame, U_b is the speed of the planar flame relative to the burning gas, D_M is the Markstein diffusivity, and γ is the thermal expansion coefficient,

$$\gamma=\frac{(\rho_f-\rho_b)}{\rho_f}, \qquad (2)$$

where ρ_f is the density of the fresh mixture and ρ_b is the density of the burned gas, $\rho_f>\rho_b$. Equation (1) is asymptotically exact in the limit of small $\gamma\ll 1$. $\Lambda\{\cdots\}$ represents a linear singular nonlocal operator defined conveniently in terms of the spatial Fourier transform by

$$\Lambda:\tilde{\Phi}(k,t)\mapsto 2\pi|k|\tilde{\Phi}(k,t),$$
$$\tilde{\Phi}(k,t)=\int_{-\infty}^{+\infty}dk\,\Phi(x,t)e^{2\pi ikx}. \qquad (3)$$

Λ is responsible for the Darrieus-Landau instability [2,3].

Direct numerical simulations for Eq. (1) performed in Ref. [4] show that even when the initial conditions are chosen to be smooth, the cusps develop on the flame interface as time increases. When the integration domain is wide enough, the secondary randomlike subwrinkles arise on the interface. Experimental studies reported in Ref. [5] show that under usual experimental conditions the wrinkling process is accompanied by the flame speed enhancement undergoing an acceleration in time $\propto t^{3/2}$.

Numerous analytical investigations devoted to Eq. (1) and to its modified version pertinent to an outward propagating flame display that in the limit of long times the local flame dynamics is driven by the large-scale geometry [6–8]. Exact solutions of Eq. (1) can be obtained in principle by using the pole decomposition technique [9–11]. For such pole solutions, Eq. (1) formally reduces to a finite set of ordinary differential equations (ODE's) which describe the motion of the poles in the complex plane. These poles are interpreted to be related to the cusps observed in physical space. However, numerical and analytical results demonstrate convincingly that the solutions of the ODE's do not resemble those obtained from the direct numerical integration of Eq. (1). In particular, the number of wrinkles obtained from the ODE's is independent of time and the corresponding (mean) expansion of the front is much slower than the $t^{3/2}$ power law.

In Refs. [12] and [13], it was argued that the inconsistencies with the pole decomposition method lie in the stability of the exact pole solutions. The initial value problem of the linearized PDE about a pole solution has been solved numerically; as a result they concluded that pole solutions are unstable for large γ. Consequently, they are not observed in experiments.

It was conjectured in Ref. [14] that nonlinearity alone is not enough to meet the experimental observations and that the results of the spectral numerical integrations is due to computational noise. In Ref. [14] a model had been developed, where pseudorandom forcing is included. It is shown that many broad-banded exciting fields indeed lead to the rapid spawning of wrinkles.

The linear stability of the pole decomposition solutions was discussed in Ref. [15] in detail. The exact analytical expressions for the eigenvalues and eigenfunctions have been constructed. Based on these expressions, in Ref. [15], they demonstrate that for any value of the parameter γ there exists the only asymptotically stable solution with the largest possible (for this particular value of γ) number of pole N_γ. As the parameter γ increases, the equilibrium states of the PDE undergo a cascade of bifurcations. In this way the new solution with N poles gains stability while the former one with $N-1$ poles becomes unstable. However, the nonlinear

*Email address: volchen@cpt.univ-mrs.fr

stability and dynamics of cusps still remain an important open question within such an approach [15].

In the present paper, we consider a pseudorandom forced analogue of the equation governing the slope function dynamics of the wrinkled flames propagated through an infinitely wide channel by the field-theory methods. We use the pseudorandom forcing on the one hand as an origin of the spawning of wrinkles, on the other hand as a ground on application of the quantum-field-theory formalism. From the very outset, we should stress that we address the asymptotic solutions of the *stochastic problem* based on the slope function equation, but not the exact solutions of this equation.

We demonstrate that the main contribution to the statistics of slopes is given by a coupled field-force configurations—the *instantons*. These configurations are related directly to a very short-time (practically instant) kinks between metastable *ground states* incident to different numbers of poles.

The paper is organized as follows. In Sec. II, we formulate the stochastic problem for the equation governing the slope function dynamics of advancing flame fronts. These fronts usually either form fractal objects with contorted and ramified appearance or they wrinkle producing self-affine fractals characterized with some critical exponent [16]. The latter fact motivates the analysis of the problem from the point of view of critical phenomena theory, which is given in Sec. III.

We must say that in the actual problem, the renormalization-group technique (which has proved itself so well in the fully developed turbulence theory; the action there functional resembles the action of the present theory [17], [18]) is ultimately ineffective since, obviously, the regime of critical scaling is not attained. One can hardly use the concept of critical dimensions for the actual quantities.

The examples of successful application of the saddle point calculations to the Burger's equation [19] and to the description of intermittency phenomenon in turbulence [20] have been given recently. These papers have inspired us to employ this technique in the problem of wrinkling flame fronts. The infinite set of instantonlike solutions we have found is dramatically dissimilar to those computed in Refs. [19] and [20].

In Sec. IV, we construct the statistical theory of wrinkles based on the action functional relevant to the actual stochastic problem.

The minimization of action discussed in Sec. V requires that the field and force be coupled in some particular configurations. We also illustrate the instanton mechanism of poles generation for the particular two poles initial configuration. The process keeps repeating itself as time increases. We then conclude in the last section.

II. THE STOCHASTIC PROBLEM FOR THE EQUATION GOVERNING THE SLOPE FUNCTION DYNAMICS

The stochastic problem for the equation governing the slope function dynamics of the flame front, $u(x,t) = \partial_x \Phi(x,t)$, with a Gaussian distributed pseudorandom force included on the right-hand-side reads as follows:

$$\partial_t u + U_b u \partial_x u = D_M \partial_x^2 u + \tfrac{1}{2}\gamma U_b \Lambda\{u\} + f. \tag{4}$$

The pair-correlation function for f is taken in the form

$$\langle f(x,t) f(x',t') \rangle = D_f(x-x')\,\delta(t-t'), \tag{5}$$

in which the function $D_f(x-x')$ is supposed to be an even smooth ''bell''-shaped function of x. To be specific, we take it in the form

$$D_f(x) = \frac{D_0}{\pi}\,\frac{m}{x^2+m^2} \tag{6}$$

decaying at the rate m and turning into $D_0\delta(x)$ as $m\to 0$, where D_0 is a constant.

Equation (4) is similar to the Burger's equation except the singular term, $\propto \gamma U_b \Lambda\{u\}$. The homogeneous equation with no external forcing (4) is considered in Ref [9] in detail. In particular, it was shown that it possesses a pole decomposition, i.e., it allows a countable number of uniform solutions,

$$u(x,t) = -2\nu \sum_{i=-N}^{N} \frac{1}{x - z_i(t)}, \tag{7}$$

in which z_i's are poles in the complex plane (coming in complex-conjugate pairs) moving according to the laws of motion of poles

$$\dot{z}_i = -2\nu \sum_{i\neq j} \frac{1}{z_i - z_j} - i\gamma U_b\,\mathrm{sgn}[\mathrm{Im}(z_i)], \tag{8}$$

where Im denotes the imaginary part of a pole. One can derive easily the corresponding steady ($\dot{z}_i = 0$), solution of the Sivashinsky equation for the simplest configurations concerning the minimal number of poles. For example, for two poles the only steady solution is given by

$$u^{(2)}(x) = -\frac{4D_M x}{x^2 + D_M^2}, \tag{9}$$

and there are two possible four-pole steady configurations,

$$u^{(4)}(x) = \pm \frac{4D_M(\pm 2x^3 + 27\sqrt{2}iD_M^3 + 9\sqrt{2}iD_M x^2)}{-x^4 \pm 54\sqrt{2}iD_M^3 x \pm 6\sqrt{2}iD_M x^3 + 81D_M^4}. \tag{10}$$

We consider large-scale asymptotic solutions pertinent to the field theory (11) undergoing a sequence of kinks between different metastable *ground states* of the type (7).

To construct the solutions with spawning wrinkles, we exploit the exact correspondence between an arbitrary stochastic dynamical problem with the Gaussian distributed random force and a quantum-field-theory [21]. A short and elegant proof had been given in Ref. [22]. The stochastic problem (4) and (5) is completely equivalent to the field theory of two fields with an action functional

$$S[u,w]=\tfrac{1}{2}\int dt\,dx\,dy\,w(x,t)D_f(x-y,t)w(y,t) - i\int dt\,dx\,w\biggl(u_t+U_b u u_x - D_M u_{xx} - \frac{1}{2}\gamma U_b \Lambda\{u\}\biggr). \tag{11}$$

Here, $w(x,t)$ is the auxiliary field, which comes into play instead of the random force f. $w(x,t)$ determines the response functions of the system, for instance, the linear-response function is $\langle uw\rangle$.

III. THE ANALYSIS OF THE ACTION FUNCTIONAL FROM THE CRITICAL PHENOMENA THEORY POINT OF VIEW

In this section, we present a brief analysis of a quantum-field-theory specified by the action functional (11) from the critical phenomena theory point of view. Such an outline appears to be important since it would shed light on a question concerning the existence of a critical regime in the wrinkling flame-front propagation problem, i.e., that for any correlation function of the theory (11) there is a definite stable large scale long-time asymptotics.

We also stress the dramatic difference between the stochastic theory of turbulence (the Navier-Stokes equation with a random forcing included [17]) and the actual problem. From the point of view of the critical phenomena theory, problems (4) and (5) are formulated erroneously.

Namely, interested in the long-time large scale asymptotics behavior of correlation functions, one has to omit the term $\propto D_M k^2$ (in the momentum Fourier space) from the action (11) in benefit to $\propto \gamma U_b|k|$. However, in this case, there are infinitely many Greens functions that have singularities with respect to a general dilatation of variables, i.e., such a theory cannot be renormalized.

One can also investigate a theory in which the both terms are included simultaneously. To our knowledge, a model where the concurrence between two terms (in the momentum Fourier space) $\propto k^2$ and $\propto |k|^{2-2\alpha}$ $(0<\alpha<1/2)$ was considered first in Ref. [23] in the framework of the renormalization-group approach. It is shown that up to the value $\alpha_c<1/2$, a regular expansion in α and ε (the deviation of the space dimensionality from its logarithmic value) can be constructed and then summed over by the standard renormalization-group procedure. The critical indices of all quantities are still fixed on their Kolmogorov's values: the critical dimensions of time $\Delta_t^K=-2/3$ and velocity $\Delta_v^K=-1/3$ (the dimension of x is taken by definition as $\Delta_x=-1$) which are well known in the fully developed turbulence theory (see Ref. [17] for a review).

However, for $\alpha>1/2$, and for the particular case of $\alpha_r=1/2$, which we are interested in, the renormalization-group method fails. The matter is in new additional singularities, which spawning in the correlation functions of the field u as $|k|\to 0$. Such singularities cannot be handled by the renormalization group in principle since they do not relate to a general scaling with respect to dilatation of variables.

The summation of the leading infrared $(|k|\to 0)$ singularities of correlation functions can be done by an infrared perturbation theory. If we limit ourselves to the functions $u_z(x,t)$ which have poles $\{z(t)\}$ in the complex plane, then the action of the singular operator Λ is reduced to a first-order derivative operator

$$\Lambda\{u_z\}=i\,\mathrm{sgn}(\mathrm{Im}[z(t)])\,\partial_x u_z(x,t), \tag{12}$$

see Ref. [9]. Then, in the momentum-frequency representation, the term with Λ can be taken into account as a small shift of frequency $(\propto\gamma)$,

$$\omega\to\omega-i\gamma U_b|k|. \tag{13}$$

The infrared perturbation theory results from the expansion of $\exp S$ over nonlinearities. The corresponding diagram technique coincides with the diagram technique of Wyld [24]. The lines in the diagrams are associated with the bare propagators, in the Fourier space,

$$G_{uw}=G_{wu}^{*}=\frac{1}{-i(\omega-i\gamma U_b|k|)+D_M k^2} \tag{14}$$

and

$$G_{uu}=\frac{1}{-i(\omega-i\gamma U_b|k|)+D_M k^2}\times D_f(k)\frac{1}{i(\omega-i\gamma U_b|k|)+D_M k^2}, \tag{15}$$

where $D_f(k)$ is the momentum representation of Eq. (6). For any correlation function, this diagram technique gives an infrared representation, which is naturally consistent with the γ expansion and is well defined for small values of the parameter γ.

We are not going to discuss the application of the infrared perturbation theory to the actual problem in detail. Here, we conclude that for any pseudodifferential operator $\propto|k|^{2-2\alpha}$, $0<\alpha<1/2$, the model of the type (11) has a critical regime with the critical indices fixed at their Kolmogorov's values (see Ref. [23]). However, for $\alpha\geqslant 1/2$, the stability of asymptotics is still an important open question if γ is large.

We conclude this section with a remark on the Kolmogorov critical dimension of time in the context of the problem in question.

The mean-squared distance of propagating flame front $R^2(t)$ can be expressed naturally via the linear-response function mentioned at the end of the previous section as follows:

$$R^2(t)=\int dx\,x^2\langle u(x,t)w(x,0)\rangle. \tag{16}$$

The requirement that each term of the action functional be dimensionless (with respect to x and t separately) leads to the power counting relation for the product, $\Delta[uw]=d$, where d is the space dimensionality. Therefore, the power of the linear-response function is

$$\Delta[\langle uw\rangle]=d-d=0. \tag{17}$$

We note that Eq. (17) is still valid whether a critical regime is attained or not.

Following a tradition, we accept the natural normalization condition that $\Delta[x]=-1$, then

$$\Delta[R^2]=-2. \tag{18}$$

On the other hand, the Kolmogorov's critical dimension of time $\Delta_t^K=-2/3$ means that

$$R^2\propto t^3 \tag{19}$$

(i.e., $R\propto t^{3/2}$). One can see that the Kolmogorov critical regime, if it has a place in the theory (11), would lead to the power-law spectrum $t^{3/2}$.

IV. STATISTICS OF SLOPES OF THE ADVANCING FLAME FRONT

We are going to discuss the saddle-point configurations of Eq. (11) which can provide us with a detailed description of the mechanism of wrinkles generation on the propagating flame-front surface. For future purposes, it would be convenient to perform consequently the rescaling of fields in Eq. (11),

$$u\to\frac{u}{U_b},\quad w\to U_b w, \tag{20}$$

such that the parameter U_b is removed from the nonlinear term wuu_x, and then another rescaling,

$$u\to\frac{u}{\gamma},\quad U_b\to\frac{U_b}{\gamma}. \tag{21}$$

As a result of such a simple transformation, we observe that the parameter of thermal expansion γ, which we assume to be small, plays the formal role of $\hbar$ in quantum-field-theory:

$$S\to\frac{1}{\gamma}S. \tag{22}$$

The correlation functions of the basic field u are then given by the functional integral

$$G_n(x_1,t_1;x_2,t_2;\ldots,x_n,t_n)=\int\mathcal{D}u\,\mathcal{D}w\,u(x_1,t_1)u(x_2,t_2),\ldots,u(x_n,t_n)\times\exp\left(-\frac{1}{\gamma}S\right), \tag{23}$$

and can be derived naturally by means of a generating functional, which has been introduced first in Ref. [25] and then employed in Refs. [19] and [20]

$$\mathcal{Z}(\lambda)\equiv\left\langle\exp\left(i\int dt\,dx\,\lambda u\right)\right\rangle=\int\mathcal{D}u\,\mathcal{D}w\exp\left(\frac{1}{\gamma}\left\{-S+i\int dt\,dx\,\lambda u\right\}\right). \tag{24}$$

The coefficients of the expansion of $\mathcal{Z}$ in λ are the correlation functions (23).

There are no general methods to compute such a functional integral exactly. The straightforward perturbative approach is to expand the exponential in the functional integral (24) in powers of the nonlinear term wuu_x. However, since we are interested in nonperturbative effects, it seems more natural to search for some saddle-point configurations that minimize the action functional (11), thus dominating the functional integral in a way similar to the saddle-point approximation in ordinary integrals. Such solutions are called *instantons*, and they determine the asymptotics of Eq. (24) at small $\gamma\ll 1$, which corresponds to WKB approximation in quantum-field-theory ($\hbar\ll 1$).

Another quantity that can be expressed via the generating functional (24) is the probability distribution function $\mathcal{P}(u)$ for the field u,

$$\mathcal{P}(u)=\int\mathcal{D}\lambda\,\mathcal{Z}(\lambda)\exp\left(-i\int dt\,dx\,\lambda u\right). \tag{25}$$

The behavior of $\mathcal{P}(u)$ for large u is also dominated by some saddle-point configurations of the integrand. However, these configurations are not the same for both Eqs. (24) and (25).

V. KINK SOLUTIONS OF THE FORCED EQUATION GOVERNING THE SLOPE FUNCTION DYNAMICS

In what follows we shall look for saddle-point configurations driven by the random force in terms of functions that have poles in the complex plane $u_z(x,t)$. To be specific, we observe generation of the four poles configuration from the two poles as a result of a kink. In contrast with Refs. [19] and [20], we need not introduce here a large artificial parameter to fix the saddle points dominating the functional integrals (24) and (25) since we have the inverse thermal expansion coefficient $1/\gamma$, which is naturally large.

Suggesting that the field u can be continued analytically on the complex plane except for the poles, we shall study the correlation function of the form

$$G(z)=\left\langle\exp\left[\frac{u(z)-u(z^*)}{\gamma}\right]\right\rangle \tag{26}$$

of two distinct points of the complex plane symmetrical with respect to the real axis. We suppose also that at the initial moment of time $t=0$ the field u can be depicted as a configuration of two complex-conjugated poles z and z^*. The function (26) possesses a generating property: Tailoring Eq. (26) in powers of $1/\gamma$, one obtains the ''structure functions'' for the field u. The functional Fourier transform (25) of Eq. (26) gives us the two-point probability distribution. The structure function generated by Eq. (26) is related to the same point $x=\mathrm{Re}(z)$ on the real axis.

Taking an average in Eq. (26) with respect to a functional measure, we perform an integration over all possible configurations $u(x,t)$ with the asymptote prescribed by initial two poles and all possible final multi-pole configurations. The basic symmetry of the action (11) is the Galilean invariance that reveals itself in the real transformation

$$u_a(x,t)\mapsto u[x+X_a(t),t]-a(t), \tag{27}$$

where $a(t)$ is an arbitrary function of t decreasing rapidly as $|t|\to\infty$ and $X_a(t)=\int_0^t dt'\, a(t')$. The transformation (27) defines an orbit in the functional space of u along which the result of functional averaging does not change. It follows that the integral itself is proportional to the volume of this orbit. This volume should be factorized before one can perform the saddle-point calculation (see Ref. [26]). It is appropriate to choose for the latter the ‘‘plane’’ transversal to the real axis $\mathrm{Re}(z)$ and then cancel out the real components of u that are related to each other via Eq. (27). In Eq. (26), the real contribution to u is subtracted out, so that it is very suitable for instanton calculations.

The asymptotics of small γ in Eq. (26) is dominated by the saddle-point configurations of the functional

$$\mathcal{W}[u,w,z]=\frac{u(z)-u(z^*)}{\gamma}-S[u,w], \tag{28}$$

which should satisfy the following equations obtained by varying Eq. (28) with respect to u and w:

$$u_t+uu_x-D_M u_{xx}-\frac{1}{2}U_b\Lambda\{u\} =-\frac{1}{2}\frac{iU_b}{\gamma}\int dx'\, D_f(x-x',t)w(y,t), \tag{29}$$

$$w_t+uw_x+D_M w_{xx}+\frac{1}{2}U_b\Lambda\{w\} =-\frac{i}{\gamma}\delta(t)\{\delta(x-z)-\delta(x-z^*)\}. \tag{30}$$

These equations for the saddle-point configurations are similar to those derived in Ref. [19] except the last singular term on the left-hand side. They follow from the Sivashinsky equation for the slope function (4), however they contain information on a special force configuration necessary to produce instantons also.

Indeed, the particular solutions of Eqs. (29) and (30) are dependent substantially from the initial data for u and w. Minimization of the action requires $u\to 0$, at $t\to-\infty$ and $w\to 0$, at $t\to\infty$. Obviously, any solution of Eq. (30) which is nonsingular as $t\to+\infty$ should be equal to zero at $t>0$ (since the field w feels a negative diffusivity). Following an analogy with Refs. [19] and [20], one can say that the field w propagates backwards in time starting from its initial value

$$w(t=-0)=-\frac{i}{\gamma_0}\{\delta[x-z(0)]-\delta[x-z^*(0)]\} \tag{31}$$

while it is zero at all later moments of time. Therefore, the system (29) and (30) as well as the integrals in Eq. (11) can be treated for $t<0$ only.

While propagating backward in time, Eq. (31) is a subject to a drift of the initial conditions as governed by the velocity in the Eq. (30), the smearing of the initial δ-function distributions in Eq. (31) due to diffusivity, and finally, an advection in the complex plane, in the imaginary direction towards the real axis. In the limit of no diffusivity, one can neglect the smearing in the Eq. (30). A simplified equation, which we arrive at when we drop the diffusivity term is just moving the δ-singular right-hand side of Eq. (30) around. Therefore, the solution of Eq. (30) can be expressed naturally in the form

$$w(t)=-\frac{i}{\gamma(t)}\{\delta[x-z(t)]-\delta[x-z^*(t)]\} \tag{32}$$

with the boundary conditions $\gamma(0)=\gamma_0$, $z(0)=z_0$. If one takes the diffusivity term in Eq. (30) into account (in the case of eventually small $D_M/\gamma U_b\leqslant 1$), the solution of Eq. (30) would be expressed in terms of even functions decaying as $|x|\to\infty$,

$$w(t)=-\frac{i}{\pi\gamma(t)}\left\{\frac{y(t)}{y^2(t)+[x-z(t)]^2} -\frac{y(t)}{y^2(t)+[x-z^*(t)]^2}\right\}, \tag{33}$$

where the parameter

$$y(t)=\frac{D_M}{\gamma(t)U_b} \tag{34}$$

is a size of an instanton, and $z(t)$ is its position changing with time (we have borrowed the terminology from the quantum-field-theory). As $|y(t)|\to 0$, the instanton shrinks to a point, and the solution (33) is reduced to Eq. (32).

To proceed with the Eq. (29), we rewrite it due to Eq. (12) in the form

$$u_t+uu_x-D_M u_{xx}-\frac{i}{2}U_b \mathrm{sgn}\,\mathrm{Im}[z(t)]\partial_x u_z(x,t) =-\frac{1}{2}\frac{iU_b}{\gamma}\int dx'\, D_f(x-x',t)w(y,t). \tag{35}$$

The one-dimensional equation (35) can be linearized by the Cole-Hopf transformation,

$$u(x,t)=-2D_M\frac{\psi_x(x,t)}{\psi(x,t)}, \tag{36}$$

so that we arrive at the equation

$$\psi_t - D_M \psi_{xx} - \frac{i}{2} U_b \mathrm{sgn}\, \mathrm{Im}[z(t)] \partial_x \psi(x,t)$$
$$= -\frac{1}{2} \frac{i U_b}{\gamma_0} \int dy\, D_f(x-y,t) w(y,t). \qquad (37)$$

Since $w(x,t)$ distinguishes from zero only for $t \leqslant 0$ and $D_f \propto \delta(t)$, the only nontrivial contribution into the right-hand side of Eq. (37) is given by the moment of time $t=0$. The general solution of Eq. (37) in the Fourier space reads as follows:

$$\psi_{inst}(k,t) = \frac{1}{2} \frac{i U_b}{\gamma_0} D_f(k,0) w(k,0) \delta(t), \qquad (38)$$

plus a transient process decaying rapidly as time growing, $\propto \exp[-t(D_M k^2 + U_b |k|)]$.

Now we can use Eqs. (6) and (33) to write down the right-hand side of Eq. (38) explicitly,

$$D_f(k,0) w(k,0) = 4\pi^2 i D_0 \exp[-[y(0)+m]$$
$$\times |k| - ik \mathrm{Re}[z_0]] \sin k\, \mathrm{Im}[z_0]. \qquad (39)$$

Performing an inverse Fourier transform of Eq. (38), one obtains

$$\psi_{inst} = \frac{D_0 U_b^3 \gamma_0}{2\pi} \left[\frac{1}{(D_M + \gamma_0 U_b m)^2 + \gamma_0^2 U_b^2 (x - z_0)^2} - \frac{1}{(D_M + \gamma_0 U_b m)^2 + \gamma_0^2 U_b^2 (x - z_0^*)^2} \right]. \qquad (40)$$

Finally, we arrive (Ref. [27]) at the following four poles configuration for the instanton u_{inst},

$$u_{inst} = \frac{4 D_M x\, \mathrm{Im}[z]}{[(D_M/\gamma_0 U_b + m)^2 + x^2 - \mathrm{Im}[z]^2]^2 + 4x^2 \mathrm{Im}[z]^2} \delta(t). \qquad (41)$$

For the last step of instanton computation we have to define the functions $\gamma(t)$ and $\varphi(t) \equiv \mathrm{Im}[z(t)]$ in Eq. (33) (remember that $\mathrm{Re}[z]$ is fixed) as $t<0$. We can do it by a direct substitution of Eq. (32) (in case of eventually small diffusivity) into Eq. (30). Here we note that since the instanton solution (41) exists as $t \geqslant 0$ (if one takes the transient process into account), we have to use the initial two-pole configuration instead of $u(x,t)$ in Eq. (30). As a result, we obtain the system of simplified equations

$$-\dot{\gamma}(t) = \frac{4 x D_M}{x^2 + \varphi(t)^2} \gamma(t), \qquad (42)$$

$$-\dot{\varphi}(t) = \frac{4 x D_M}{x^2 + \varphi(t)^2} \varphi(t). \qquad (43)$$

The formal solution of Eq. (42) is given by

$$\gamma(t) = \gamma_0 \exp\left(-\int_t^0 \frac{4 x D_M}{x^2 + \varphi(t')^2} dt' \right), \quad (t'<0) \qquad (44)$$

and can be computed, in principle, if one knows $\varphi(t)$. Equation (43) is equivalent to

$$\frac{x}{D_M} \ln \varphi(t) + \frac{1}{2 x D_M} \varphi^2(t) + t = C, \qquad (45)$$

which leads to

$$\varphi(t) = \varphi_0 \exp\left[-\frac{t D_M}{x} - \frac{1}{2} W\left(\frac{e^{-2tD_M/x}}{x^2} \right) \right]. \qquad (46)$$

$W(x)$ is the Lambert function that meets the equation

$$W(x) \exp W(x) = x. \qquad (47)$$

The latter equation has an infinite number of solutions for each (nonzero) value of x. W has an infinite number of branches numbered by an integer number $n \in [-\infty, \ldots, \infty]$. Exactly one of these branches is analytic at 0 (the principal branch, $n=0$). The other branches all have a branch point at 0. The principal branch is real valued for x in the range $-\exp(-1), \ldots, \infty$, while the image of $-\infty \ldots -\exp(-1)$ under $W(x)$ is the curve $-y\cot(y) + yi$, for $y \in [0, \ldots, \pi]$. For all the branches other than the principal branch, the branch cut dividing them is the negative real axis. The image of the negative real axis under the branch $W(n,x)$ is the curve $-y\cot(y) + yi$, for $y \in [2k\pi, \ldots (2k+1)\pi]$ if $k>0$ and $y \in [(2k+1)\pi, \ldots (2k+2)\pi]$ if $k<-1$. These curves, therefore, bound the ranges of the branches of W, and in each case, the upper boundary of the region is included in the range of the corresponding branch.

Each particular orbit of Eq. (46) provides a distinct solution of Eqs. (44) and (32). However, each configuration $w(x,t)$, $t<0$ which enjoys Eq. (32) is related to the same configuration $u(x,t)$, $t>0$. The value of Eq. (28) for the instanton is obviously finite, however, one can hardly compute it for each branch of $w(z,t)$.

Let us consider the principle branch of the function $\varphi(t)$ just to illustrate the idea of computation. One can check that the leading contribution to φ is accumulated around $x=0$. The asymptotic behavior of W at complex infinity and at 0 is given by

$$W(x) \sim \log(x) - \log[\log(x)] + \sum_{m,n=0}^{\infty} C(m,n) \frac{\log[\log(x)]^{(m+1)}}{\log(x)^{(m+n+1)}}, \qquad (48)$$

where $\log(x)$ denotes the principal branch of the logarithm, and the coefficients $C(m,n)$ are constants.

Restricting to the first term of the asymptote (48), one obtains

$$\varphi(t) \simeq -\varphi(0) D_M t, \quad (t<0), \qquad (49)$$

then we use Eqs. (49) and (44) to compute $\gamma(t)$ as $t<0$,

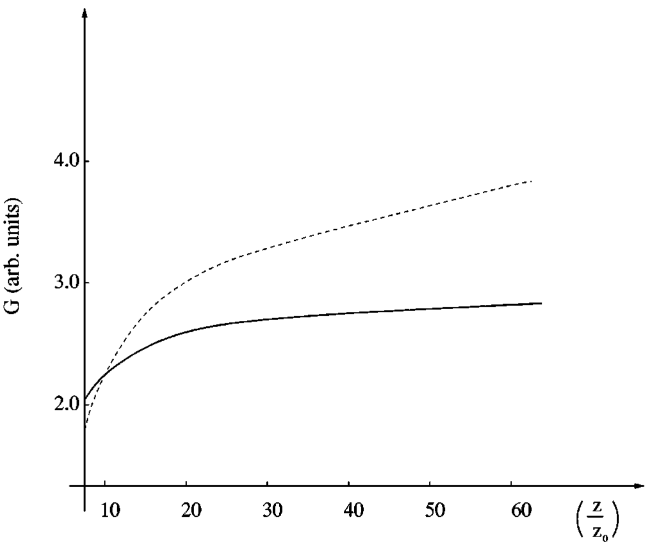

FIG. 1. The comparison of the model curve of the asymptotics prescribed by Eq. (52) (the solid line) and the curve corresponding to the nonexisting critical regime with the Kolmogorov's critical exponent (the dashed line). It is clear that the acceleration of the wrinkled flame moving through a wide channel, whose wrinkling process is driven by the pseudorandom forcing, is slower than that in the Kolmogorov's critical regime.

$$\gamma \simeq \gamma_0 \exp\left[\frac{4}{\varphi_0}\tan^{-1}\frac{\varphi_0 D_M t}{x}\right], \quad t<0. \tag{50}$$

Now it is a matter of a simple computation to find the action on the principal branch of the instanton. We collect everything together and substitute Eqs. (50), (49), (41), and (32) back to Eq. (11) to obtain

$$S_{inst} = -\frac{D_0}{m}\exp\left[-\frac{2}{\varphi_0}\tan^{-1}\varphi_0\right], \tag{51}$$

while the correlation function we have been studying is

$$G \propto \exp\left(\frac{D_0}{m}\exp\left[-\frac{2}{\varphi_0}\tan^{-1}\varphi_0\right]\right). \tag{52}$$

This simplified formula is obviously not an exact answer. It is just a leading asymptotic of $G(\mathrm{Im}[z_0])$, if $1/\gamma_0$ is a large number, $\varphi_0 \equiv \mathrm{Im}[z_0]$ is eventually small, however, not too small (since we have not taken the smearing due to diffusivity into account). In Fig. 1, we have plotted out both the model curve of the asymptotics prescribed by Eq. (52) and by the curve corresponding to the critical regime with the Kolmogorov's critical exponent for velocity $\propto k^{-1/3}$. It is clear that the acceleration of flame moving through a wide channel, whose wrinkling process is driven by the pseudorandom forcing, is slower than that in the Kolmogorov's critical regime.

Due to a specific property of the action (11), the contribution from fluctuations of δw and δu up to second order will be zero. To find higher-order corrections to Eq. (52), one has to consider at least the third-order terms. We investigate this problem in future publications.

VI. DISCUSSION AND CONCLUSIONS

The study performed in this paper confirms that the stochastic model for the equation governing the slope dynamics of flames propagating in a wide channel in the regime of well-developed hydrodynamic instability demonstrates the self-fractalization of the flame front. The mechanism consists of successive instabilities through which the interface becomes more and more wrinkled as time increases. The main contribution to the statistics of slopes of a wrinkling flame front propagating in a wide channel is due to a coupled field-force configurations, which are found to be responsible for the birth and growth of wrinkles. The relevant asymptotic behavior has a transient nature, and it is organized in a sequence of kinks between pole solutions for which the number of poles is constant. The acceleration of the mean front moving through an infinitely wide channel is clearly due to successive births of poles.

We have shown that if the critical regimes existed in the model, the $t^{3/2}$ acceleration would be a consequence of the Kolmogorov's scaling with the critical dimensions of time $\Delta_t^K = -2/3$, which is well known in the fully developed turbulence theory.

Provided the critical regime in the model of wrinkling flames exists, then it means that each correlation function in the model has a definite stable long-time large scale asymptotics, which does not depend on the particular sequence of kinks. If one replaces the nonlocal operator Λ with some pseudodifferential operator $\propto |k|^{2-2\alpha}$, $0<\alpha<1/2$, then, as it was shown in Ref. [23], the model of the type (11) has a critical regime with critical indices fixed at their Kolmogorov's values. However, for $\alpha \geqslant 1/2$, the stability of asymptotics is still an important open question if γ is large.

We have used the saddle-point calculations assuming the inverse thermal expansion coefficient $1/\gamma$ as a large parameter. As a result, we construct an infinite family of instanton solutions numbered by $n \in \mathbb{Z}$. Each instanton determined by one of the branches of the Lambert function $W(x)$ has a unique behavior as $t<0$, however, all instantons are indistinguishable as $t>0$.

The crucial problem of the developed technique is of contribution to the action from the fluctuations against the instanton background. The general analysis of the set of instantons, which we have found, will be published in a forthcoming paper.

We also note that the developed technique needs an essential modification to be applied for the flames propagating through finite-width channels. Actually, the new dimensional parameter L, the width of the channel, would create a new massive term into the action functional (11). This can change the results completely. On physical grounds one may expect that the flame speed will saturate as $t \to \infty$ in the finite-width channel.

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